CARINGBAH HIGH SCHOOL

2009

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

Reading time - 5 minutes

Working time - 3 hours

Write using black or blue pen.

Board-approved calculators may be used.

A table of standard integrals is provided at the back of this paper.

Total marks - 120

Attempt Questions 1 - 10

All questions of equal value.

All necessary working should be shown in every question.

Question 1 (12 marks) **Marks**

(a) Factorise $2x^2 + 7x - 15$ **2**

(b) Solve $x + 5 > 9$ and graph the solution on the number line. **2**

(c) Find a primitive of $x^2 - 3$ **2**

(d) Rationalise the denominator: $\dfrac{1}{\sqrt{3} - 2}$ **2**

(e) Solve the pair of simultaneous equations **2**

$$2x + y = 5$$
$$x - 2y = -5$$

(f) Find the exact value of $\sin(\dfrac{\pi}{3}) + \sin(\dfrac{3\pi}{4})$ **2**

Question 2 (12 marks) Start a new page.

(a) Differentiate with respect to x:

(i) $(3x-4)^8$

(ii) $x^2 \sin x$

(iii) $\dfrac{\log_e x}{x}$

(b) Find the sum of the first 25 terms of t

(c) (i) Find $\displaystyle\int \frac{2x}{x^2+3}\,dx$

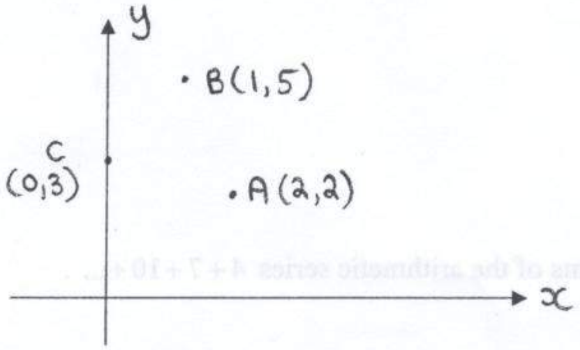
(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sin 2x\,dx$

Question 3 (12 marks) Start a new page.

(a) (i) Differentiate $\sin^2 x$.

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \sin x \cos x \, dx$.

(b)



The diagram shows three points $A(2,2)$, $B(1,5)$ and $C(0,3)$.

(i) Find the length of AB (leaving your answer in surd form).

(ii) Find the gradient of AB.

(iii) Show that the equation of line AB is $3x + y - 8 = 0$

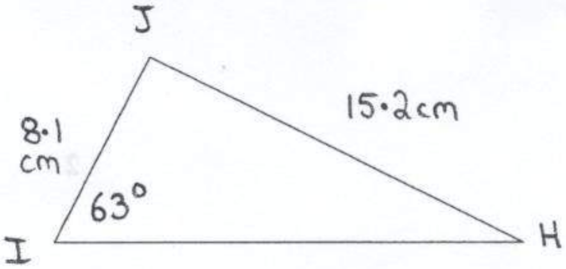
(iv) Find the perpendicular distance of C to AB.

(v) Hence, or otherwise, find the area of the triangle ABC.

(vi) Write down the gradient of the line perpendicular to AB.

Question 4 (12 marks) Start a new page.

(a) Use the sine rule to find the size of $\angle H$, correct to the nearest degree: **2**



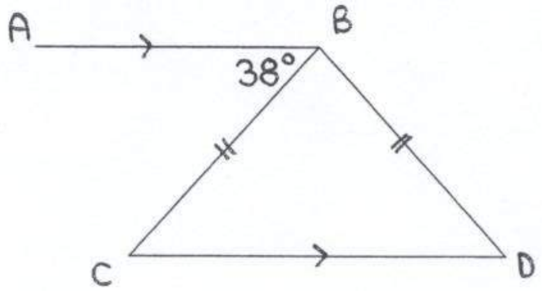
(b) A prestige car, the SHOWOFF 92B, costs \$73 000 brand new. Its value depreciates 15% per year.

(i) What is the car's value one year after it is purchased? **1**

(ii) An automobile museum wants to buy a SHOWOFF 92B but can afford to pay no more than \$20 000. What is the least number of years the museum must wait to purchase one? **2**

(c) If $\sin A = -\frac{1}{3}$ and $\tan A \leq 0$, find $\sec A$. **2**

(d) In the diagram AB is parallel to CD and $BC = BD$ **3**
Giving reasons, find the size of $\angle CBD$:



(e) The gradient of a curve is given by $\frac{dy}{dx} = 1 + e^{2x}$. The curve passes through $\left(0\ ,\ 4\frac{1}{2}\right)$. What is the equation of the curve? **2**

Question 5 (12 marks) Start a new page.

(a) In the infinite geometric series $1 + x + x^2 + x^3 + \ldots$, the first term is four times the sum of <u>all</u> the terms following it.

Find the value of x.

(b) Solve $\log_{10}(x^2) + 3 = \log_{10}(x^5)$

(c) Consider the parabola $(x-2)^2 = 4y$

(i) Write down the coordinates of the vertex.

(ii) Find the coordinates of the focus.

(iii) What is the equation of the directrix?

(iv) Sketch the parabola.

(v) Calculate the area enclosed by the parabola and the x and y axes.

Question 6 (12 marks) Start a new page.

(a) The number of bacteria (P) in a colony after t minutes is given by

$$P = 1000e^{0.04t}.$$

Find:

(i) the number of bacteria when $t = 10$.

(ii) the rate at which the colony is increasing when $t = 10$.

(iii) how many minutes it takes for the colony to double in size.

(b) Find all the values of θ, where $0^\circ \leq \theta \leq 360^\circ$, that satisfy the equation

$$\cos^2 \theta = \frac{3}{5}.$$

Give your answer(s) to the nearest degree.

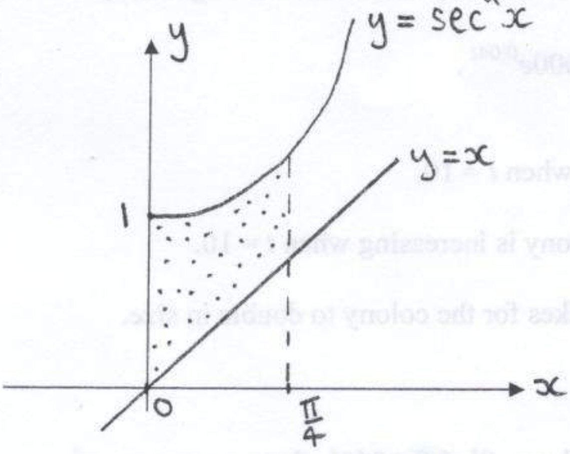
(c) Find the values of m for which $10 + 3m - m^2 < 0$

(d) The number N of cane toads in the Townsville Dam at any time t was studied over a number of years.
The study concluded that although the number of cane toads was still increasing, government measures to eradicate them were beginning to take effect.

Write down the signs (positive or negative) of $\frac{dN}{dt}$ and $\frac{d^2N}{dt^2}$.

Question 7 (12 marks) Start a new page.

(a)



The graphs of $y = \sec^2 x$ and $y = x$ between $x = 0$ and $x = \frac{\pi}{4}$ are shown on the diagram.

Calculate the area of the shaded region.

(b) Use Simpson's Rule with 5 function values to find an approximate value of $\int_1^5 x \log_e x \, dx$.

(c) The velocity of a particle, v metres per second at time t seconds, is given by

$$v = 8t - t^2$$

(i) When is the particle at rest?

(ii) When is the acceleration of the particle equal to zero?

(iii) What is the distance travelled by the particle in the first 5 seconds?

(d) The sixth term of an arithmetic sequence is 17 and the thirteenth term is 31. Find the common difference.

Question 8 (12 marks) Start a new page.

(a) Let $f(x) = x^3 - 6x^2 + 9x - 5$.

(i) Find the coordinates of the stationary points of $f(x)$ and determine their nature.

(ii) Find the coordinates of any points of inflexion.

(iii) Sketch the graph of $f(x)$ in the domain $0 \leq x \leq 5$.
[You do not have to show x-intercepts]

(b) Evaluate $\int_0^{\ln 8} e^{-2x}\, dx$

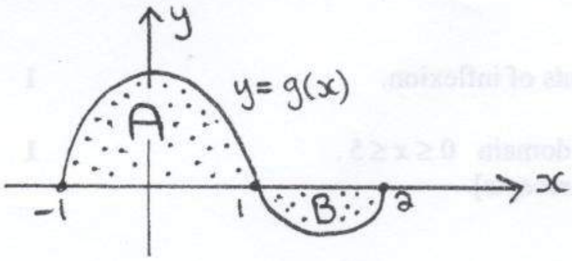
(c) (i) Draw the graphs of $y = 3\sin x$ and $y = x - 2$ on the same set of axes for $0 \leq x \leq 2\pi$.

(ii) Using these graphs, determine how many solutions the equation $3\sin x = x - 2$ has in the domain $0 \leq x \leq 2\pi$.

Question 9 (12 marks) Start a new page.

(a) Solve the equation $|2x-11| = 3x-4$ **2**

(b) The function $y = g(x)$ is sketched below. **1**



The shaded area A is 7 square units and shaded area B is 4 square units.

Evaluate $\int_{-1}^{2} g(x)\,dx$

(c) Nancy invests $\$D$ at 6% per annum compounded annually. She plans to withdraw \$4000 at the end of each year, for the next five years, to help with university expenses.

(i) Write down an expression for the amount $\$A_1$ remaining in the account following the withdrawal of the first \$4000. **1**

(ii) Find an expression for the amount $\$A_2$ remaining in the account after the second withdrawal. **1**

(iii) Calculate the amount $\$D$ that Nancy needs to invest if the account balance is to be $\$0$ at the end of five years. **3**

(d) (i) Show that the perpendicular distance of the point (3,1) from the line $mx - y = 0$ is: **1**

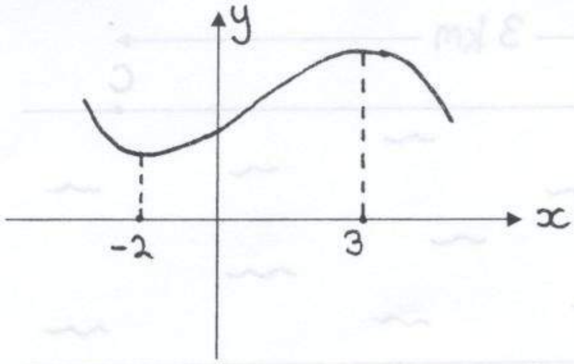
$$\frac{|3m-1|}{\sqrt{m^2+1}}$$

(ii) If $mx - y = 0$ is a tangent to the circle $(x-3)^2 + (y-1)^2 = 4$ find the 2 possible values of m in exact (surd) form. **3**

Question 10 (12 marks) Start a new page.

(a) Find the gradient of the normal to the curve $y = x^2 - x$ at the point $(3,6)$.

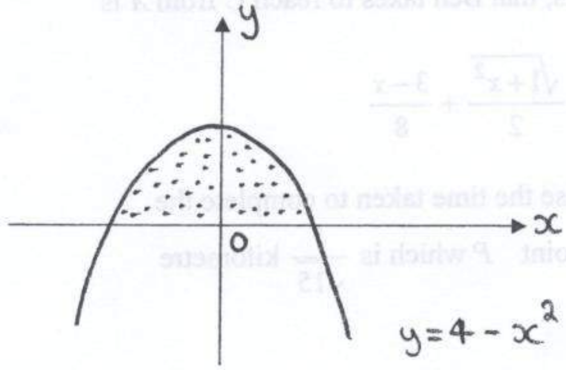
(b) The diagram shows the graph of a certain function *f(x)*.



(i) Copy this diagram.

(ii) On the <u>same</u> set of axes, draw a sketch of the derivative $f'(x)$

(c)

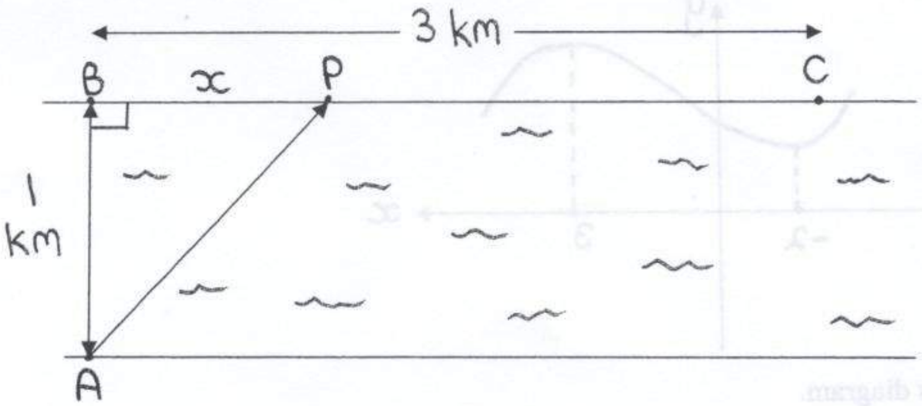


The shaded region lying between the curve $y = 4 - x^2$ and the x axis is rotated about the x axis

Find the volume of the solid of revolution so formed.

Question 10 continues on page 12.

(d) The diagram shows a straight section of river, one kilometre wide. Ben is at point A on one bank and wishes to reach a point C on the opposite bank. The point B is directly opposite A and the distance from B to C is 3 kilometres.



Ben can swim at 2 *km/h* and jog at 8 *km/h*. He plans to swim in a straight line to point P on the opposite bank then jog directly from P to C.

Let the distance from B to P be x kilometres.

(i) Show that the time t, in hours, that Ben takes to reach C from A is **1**

$$t = \frac{\sqrt{1+x^2}}{2} + \frac{3-x}{8}$$

(ii) Show that for Ben to minimise the time taken to complete the journey he should swim to point P which is $\frac{1}{\sqrt{15}}$ kilometre from B. **4**

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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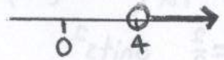
1. (a) $(2x-3)(x+5)$

(b) $x > 4$



(c) $\frac{1}{3}x^3 - 3x$

(d) $\frac{1}{\sqrt{3}-2} \times \frac{\sqrt{3}+2}{\sqrt{3}+2} = \frac{\sqrt{3}+2}{-1}$

$= -\sqrt{3} - 2$

(e) ① × 2 : $4x + 2y = 10$ – ③

③ + ② : $5x = 5$

$x = 1$

sub $x = 1$ into ① : $y = 3$

solution $(1, 3)$

(f) $\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} = \frac{\sqrt{6}}{2\sqrt{2}} + \frac{2}{2\sqrt{2}}$

$= \frac{\sqrt{6}+2}{2\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{2}$

2 (a) (i) $8(3x-4)^7 \times 3 = 24(3x-4)^7$

(ii) $u = x^2$ $v = \sin x$

$u' = 2x$ $v' = \cos x$

$y' = vu' + uv'$

$= \sin x . 2x + x^2 . \cos x$

$= x(2\sin x + x\cos x)$

(iii) $u = \log_e x$ $v = x$

$u' = \frac{1}{x}$ $v' = 1$

$y' = \frac{vu' - uv'}{v^2}$

$= \frac{x . \frac{1}{x} - \log_e x . 1}{x^2}$

$= \frac{1 - \log_e x}{x^2}$

(b) $a = 4$, $d = 3$, $n = 25$ $S_n = \frac{n}{2}[2a + (n-1)d]$

$= \frac{25}{2}[8 + 24 \times 3]$

$= 1000$

(c) (i) $\ln(x^2+3) + c$

(ii) $-\frac{1}{2}[\cos 2x]_0^{\frac{\pi}{3}} = -\frac{1}{2}[\cos\frac{2\pi}{3} - \cos 0]$

$= -\frac{1}{2}[-\frac{1}{2} - 1]$

$= \frac{3}{4}$

3 (a) (i) $\frac{d}{dx}(\sin x)^2 = 2\sin x . \cos x$

(ii) $\frac{1}{2}[\sin^2 x]_0^{\frac{\pi}{2}} = \frac{1}{2}[\sin^2\frac{\pi}{2} - \sin^2 0]$

$= \frac{1}{2}[1^2 - 0^2]$

$= \frac{1}{2}$

(b) (i) $AB = \sqrt{(1-2)^2 + (5-2)^2}$

$= \sqrt{10}$ units^2

(ii) $m_{AB} = \frac{5-2}{1-2}$

$= -3$

(iii) equn AB: $y - 2 = -3(x-2)$

$y - 2 = -3x + 6$

$3x + y - 8 = 0$

(iv) $d = \left| \frac{3 \times 0 + 3 - 8}{\sqrt{3^2 + 1^2}} \right|$

$= \left| \frac{-5}{\sqrt{10}} \right|$

$= \frac{5}{\sqrt{10}}$ or $\frac{\sqrt{10}}{2}$

(v) area $\Delta ABC = \frac{1}{2} \times AB \times d$

$= \frac{1}{2} \times \sqrt{10} \times \frac{\sqrt{10}}{2}$

$= 2\frac{1}{2}$ units^2

(vi) $m = \frac{1}{3}$

4. (a) $\frac{\sin H}{8 \cdot 1} = \frac{\sin 63°}{15 \cdot 2}$

$\sin H = \frac{8 \cdot 1 \sin 63°}{15 \cdot 2}$

$H \doteqdot 28°$

1

4 (b) (i) Value $= 73000 \times 0.85$
$= \$62\,050$

(ii) Solve $73\,000 \times (0.85)^n = 20\,000$

$(0.85)^n = 0.274$

$\ln (0.85)^n = \ln 0.274$

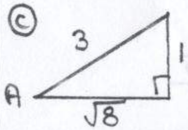
$n \ln 0.85 = \ln 0.274$

$n = \dfrac{\ln 0.274}{\ln 0.85}$

$= 7.966$

$\therefore$ after 8 years

(c)



$\sec A = \dfrac{3}{\sqrt{8}}$ [A is in 4th quad.]

$= \dfrac{3\sqrt{2}}{4}$

(d) $\angle C = 38°$ (altern. ∠s, AB∥CD)

$\therefore \angle D = 38°$ (base ∠s isos. ΔBCD)

$\therefore \angle CBD = 104°$ (∠ sum ΔBCD)

(e) $y = x + \frac{1}{2}e^{2x} + c$

sub $(0, 4\frac{1}{2})$: $4\frac{1}{2} = 0 + \frac{1}{2}e^0 + c$

$c = 4$

$\therefore y = x + \frac{1}{2}e^{2x} + 4$

5 (a) $1 = 4(x + x^2 + x^3 + \dots)$

$1 = 4 \times \dfrac{x}{1-x}$ using $S_\infty = \dfrac{a}{1-r}$ with $a = x$, $r = x$

$\dfrac{1}{4} = \dfrac{x}{1-x}$

$4x = 1 - x$

$5x = 1$

$x = \frac{1}{5}$

(b) $2\log_{10} x + 3 = 5\log_{10} x$

$3 = 3\log_{10} x$

$1 = \log_{10} x$

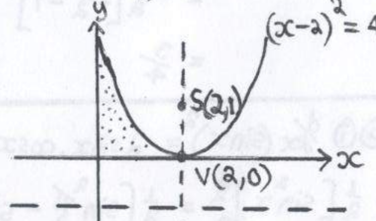
$x = 10$

(c) (i) $V(2,0)$ [of form $(x-h)^2 = 4a(y-k)$]

(ii) $a = 1$: $S(2,1)$

(iii) directrix $y = -1$

(iv)



(v) $A = \displaystyle\int_0^2 \frac{1}{4}(x-2)^2\,dx$

$= \frac{1}{12}\left[(x-2)^3\right]_0^2$

$= \frac{1}{12}\left(0^3 - (-2)^3\right)$

$= \frac{2}{3}$ units²

6. (a) (i) $P = 1000e^{0.04 \times 10}$

$= 1492$

(ii) $\dfrac{dP}{dt} = 1000e^{0.04t} \times 0.04$

$= 40e^{0.04t}$

when $t = 10$: $\dfrac{dP}{dt} = 40e^{0.4}$

$= 59.7$ bacteria/min

(iii) $2000 = 1000e^{0.04t}$

$2 = e^{0.04t}$

$\ln 2 = \ln e^{0.04t}$

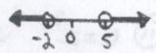
$\ln 2 = 0.04t \ln e \Rightarrow t = \dfrac{\ln 2}{0.04} \doteqdot 17$ min

(b) $\cos\theta = \pm\sqrt{3/5}$ [BASIC ∠: $\cos 39° \doteqdot \sqrt{\frac{3}{5}}$]

$\theta = 39°, 141°, 219°, 321°$

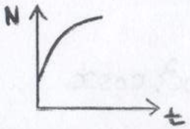
(c) $m^2 - 3m - 10 > 0$

$(m-5)(m+2) > 0$



$m < -2$ or $m > 5$

(d)



$\dfrac{dN}{dt} > 0$ [incr.]

$\dfrac{d^2N}{dt^2} < 0$ [concave down]

7. (a) $A = \int_0^{\frac{\pi}{4}} \sec^2 x - x \, dx$

$= \left[\tan x - \frac{1}{2}x^2\right]_0^{\frac{\pi}{4}}$

$= \left[\tan\frac{\pi}{4} - \frac{1}{2}\left(\frac{\pi}{4}\right)^2\right] - \left[\tan 0 - 0\right]$

$= 1 - \frac{\pi^2}{32}$ units^2

(b)

x	1	2	3	4	5
y	0	1·386	3·296	5·545	8·047
	y_0	y_1	y_2	y_3	y_4

$h = 1$

$\int_1^5 x \log_e x \, dx = \frac{h}{3}\left[y_0 + y_4 + 4(y_1 + y_3) + 2(y_2)\right]$

$= \frac{1}{3}\left[0 + 8{\cdot}047 + 4(1{\cdot}386 + 5{\cdot}545) + 2 \times 3{\cdot}296\right]$

$\doteqdot 14{\cdot}1$

(c) (i) $8t - t^2 = 0$

$t(8-t) = 0$

$t = 0, 8$ s

(ii) acc. $= 8 - 2t$

$\therefore$ acc $= 0$ when $t = 4$ s

(iii) Since particle does not stop in 1st 5s, it does not change direction in 1st 5s.

$x = 4t^2 - \frac{1}{3}t^3 + c$

when $t=0$: $x = c$

when $t=5$: $x = 58\frac{1}{3} + c$

$\therefore$ distance travelled $= 58\frac{1}{3}$ m.

(d) use $T_n = a + (n-1)d$

$\therefore a + 5d = 17$ (1)

$a + 12d = 31$ (2)

(2) − (1): $7d = 14$

common difference $d = 2$

8. (a) (i) $f'(x) = 3x^2 - 12x + 9$

Stationary points $f'(x) = 0$: $3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3, 1$

$\therefore$ stat. points: $(1,-1)$ and $(3,-5)$.

$f''(x) = 6x - 12$

$(1,-1)$: $f''(1) = -6 < 0$ $\therefore$ max. turn. point

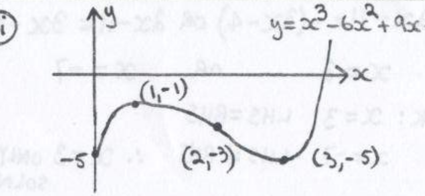
$(3,-5)$: $f''(3) = 6 > 0$ $\therefore$ min. turn. point

(ii) points inflex. $f''(x) = 0$: $6x - 12 = 0$

$x = 2$

$\therefore$ point inflex. $(2,-3)$

(iii)



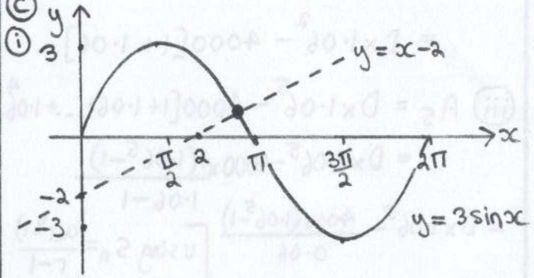
(b) $-\frac{1}{2}\left[e^{-2x}\right]_0^{\ln 8} = -\frac{1}{2}\left[e^{-2\ln 8} - e^0\right]$

$= -\frac{1}{2}\left[e^{\ln 8^{-2}} - 1\right]$

$= -\frac{1}{2}\left[8^{-2} - 1\right]$

$= \frac{63}{128}$

(c) (i)



$y = x - 2$ has intercepts $(2,0)$ and $(0,-2)$

(ii) $y = 3\sin x$ and $y = x - 2$ have 1 point of intersection in domain $0 \leq x \leq 2\pi$

$\therefore 3\sin x = x - 2$ has 1 solution in domain $0 \leq x \leq 2\pi$

9 (a) $2x-11=-(3x-4)$ OR $2x-11=3x-4$

$x=3$ OR $x=-7$

CHECK: $x=3$ LHS = RHS

$x=-7$ LHS $\neq$ RHS $\therefore x=3$ ONLY SOLN.

(b) $\int_{-1}^{2} g(x)\,dx = 7+(-4) = 3$

(c) (i) $A_1 = D\times 1{\cdot}06 - 4000$

(ii) $A_2 = A_1 \times 1{\cdot}06 - 4000$

$= [D\times 1{\cdot}06 - 4000]\times 1{\cdot}06 - 4000$

$= D\times 1{\cdot}06^2 - 4000\times 1{\cdot}06 - 4000$

$= D\times 1{\cdot}06^2 - 4000[1+1{\cdot}06]$

(iii) $A_5 = D\times 1{\cdot}06^5 - 4000[1+1{\cdot}06+\ldots+1{\cdot}06^4]$

$= D\times 1{\cdot}06^5 - 4000\times \dfrac{1(1{\cdot}06^5-1)}{1{\cdot}06-1}$

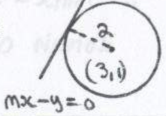
$= D\times 1{\cdot}06^5 - \dfrac{4000(1{\cdot}06^5-1)}{0{\cdot}06}$

[using $S_n = \dfrac{a(r^n-1)}{r-1}$ with $a=1, r=1{\cdot}06, n=5$]

Solving $A_5 = 0$: $D = \$16849{\cdot}46$

(d) (i) Use $d = \left|\dfrac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$ with $(x_1,y_1)\Rightarrow(3,1)$

(ii) If $mx-y=0$ is a tangent distance to $(3,1)$ is 2.



$\left|\dfrac{3m-1}{\sqrt{m^2+1}}\right| = 2$ ie. $\dfrac{3m-1}{\sqrt{m^2+1}} = \pm 2$

Square both sides: $\dfrac{(3m-1)^2}{m^2+1} = 4$

$9m^2-6m+1 = 4(m^2+1)$

$5m^2-6m-3=0$

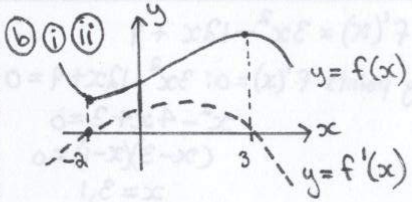
$m = \dfrac{6\pm\sqrt{96}}{10} = \dfrac{3\pm 2\sqrt{6}}{5}$

10. (a) $y' = 2x-1$

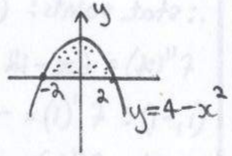
At $(3,6)$: $m_{TAN} = 2\times 3 - 1 = 5$

$\therefore m_{NORM} = -\frac{1}{5}$

(b) (i) (ii)



(c) x-ints: $x=\pm 2$



$V = \pi\int_{-2}^{2}(4-x^2)^2\,dx$

$= 2\pi\int_{0}^{2} 16-8x^2+x^4\,dx$

$= 2\pi\left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2$

$= 2\pi\left[\left(32-\frac{64}{3}+\frac{32}{5}\right)-(0-0+0)\right]$

$= \dfrac{512\pi}{15}$ units^3

(d) (i) By Thm of Pythagoras: $AP = \sqrt{1+x^2}$

Also $PC = 3-x$

using time $= \dfrac{\text{dist.}}{\text{speed}}$: $t = \dfrac{\sqrt{1+x^2}}{2} + \dfrac{3-x}{8}$

(ii) $t = \frac{1}{2}(1+x^2)^{\frac{1}{2}} + \frac{3}{8} - \frac{1}{8}x$

$\dfrac{dt}{dx} = \frac{1}{2}x(1+x^2)^{-\frac{1}{2}} - \frac{1}{8}$

Stationary points: $\dfrac{dt}{dx}=0 \Rightarrow \frac{1}{2}x(1+x)^{-\frac{1}{2}} - \frac{1}{8} = 0$

$\dfrac{x}{2\sqrt{1+x^2}} = \dfrac{1}{8}$

SQUARE B.S.: $\dfrac{x^2}{4(1+x^2)} = \dfrac{1}{64}$

$64x^2 = 4+4x^2$

$\Downarrow$

$x = \dfrac{1}{\sqrt{15}}$ [take only +ve]

$\therefore$ stationary point when $x = \dfrac{1}{\sqrt{15}}$

$\dfrac{d^2t}{dx^2} = \dfrac{1}{2\sqrt{(1+x^2)^3}}$ using quotient rule.

>0 for all x.

$\therefore$ when $x = \dfrac{1}{\sqrt{15}}$, t is minimum